Structuralism and Meta-mathematics

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Outline of the Presentation

The structuralist idea

Main idea of structuralism:
“[I]n mathematics the primary subject-matter is not the individual mathematical objects but rather the structures in which they are arranged.” (Resnik 1997 p. 201)

Structuralism rejects ontological independence of mathematical objects (in contrast to conventional Platonism)
$\Rightarrow$ Relations are crucial.

The structuralist thesis

Structuralism can be formulated in terms of criteria of identity as the following structuralist thesis (ST):

(ST) Criteria of identity for the objects of mathematics must be specified exclusively in terms of the relations obtaining between them.

Here: General plausibility of structuralism (for example with respect to natural numbers) not discussed. Interest of structuralism taken for granted.

The problem / The challenge

Structuralist thesis seems implausible with respect to meta-mathematics (specifically proof-theory)

According to common wisdom: Meta-mathematics (MM) is study of mathematics by means of mathematics.

Objects of MM: symbols, formulae, axiom systems.

$\Rightarrow$ Individuated only according to their relations???

Parsons and quasi-concrete objects

Quasi-concrete objects (Parsons' notion) are objects which are themselves abstract but have concrete instantiations. Letter *types* are quasi-concrete.

Parsons: Meta-mathematics is mathematical theory whose objects are linguistic. "*These are quasi-concrete objects, and so long as they are viewed in this way the structuralist view will not hold for them.*" (Parsons 1990 p. 337)

Artificial structuralist re-interpretation of MM is of no help:
If on the original reading MM is non-structural, it remains a stumbling block

$\Rightarrow$ Structuralists must show: MM *has never been* about quasi-concrete objects.

Three possible strategies

The structuralist strategies for making sense of MM

- Notion of quasi-concreteness incoherent. (unpromising, symbol types are unproblematic)
- Signs employed in mathematics are not symbol types (which are quasi-concrete). (implausible, we use everyday letters and numerals to operate with in mathematics)
- MM is not really about linguistic objects (mathematical signs).

Only the third option seems promising.

Formal symbols vs. “letters”

Terminological distinction:

symbol types (“letters”, linguistic objects)
vs.
formal symbols (mathematical objects)

Motivation: What we call metatheory of a mathematical theory is invariant under notation switch (different “letters”).

Example metatheory of Peano arithmetic: PA formulated in language L. Could write “$'$” instead of “S”, “Ω” instead of “0”, etc. Elements of L (“formal symbols”) and concatenations thereof remain the same.

Criteria of identity for linguistic objects

Examples of ordinary symbol types (quasi-concrete) include: letters of everyday alphabet, arabic numerals, etc.

Individuation according to everyday standards, taught to children.

Does it matter what counts as an instantiation? YES.

Criteria of identity for formal symbols I

In MM formal symbols are *represented* by linguistic objects:

symbol token $\stackrel{\text{instantiate}}{\longrightarrow}$ symbol type (“letter”) $\stackrel{\text{represent}}{\longrightarrow}$ formal symbol

Question: What are criteria of identity for formal symbols?

Does it matter which letters are admissible as representations of the same formal symbol? NO.

Arbitrary letters are ok, provided one keeps order in ongoing conversation.

Criteria of identity for formal symbols II

Structuralist can argue: Criteria of identity for (concatenations of) formal symbols are as relational as those for, say, numbers.

Example: "$SS0$" is that concatenation which consists of (formal symbol) "S" at first and second and "0" at third position.

(ST) vindicated with respect to MM.

(For an axiomatization of concatenation see: Grzegorczyk, A. (2005). Undecidability without arithmetization, *Studia Logica, 79*, 163-230)

“Representation” as application

What is “representation” of formal symbols by linguistic objects (“letters”)?

Since formal symbols are mathematical and linguistic objects empirical (though still abstract): *application* of mathematics to non-mathematical domain of objects

Comparison: Points of physical space(-time) “represent” points of analytic geometry in similar fashion.

Question: What’s the point of this application?
Predict (and explain) the findings of mathematicians.

Diagnosis

Why MM wrongly seemed to pose problems for structuralism at all:

Some mathematical theories are practiced in direct orientation to one particular application
Compare: Euclidean geometry and physical space (clarification due to Hilbert and, in particular, Einstein)

$\Rightarrow$ Misguided impression: Theory seems to be *about* what it is in fact *applied* to.

Case of MM: *Applied* to, not *about* systems of linguistic objects.

Shapiro's proposal

Shapiro's 2005 proposal for how to be a structuralist about MM:

"*[A]ssertory statements about interpretations, deductions, relative consistency, and the like, are an application of the background meta-theory, perhaps the standard application.*"

On that account, there are meta-mathematical assertions which are essentially non-structural and form an application of mathematics to philosophy!

Present account: MM no more difficult to incorporate into structuralist account than other parts of mathematics.

Application not to philosophy but to predict and explain what mathematicians do and what results they come up with.

Summary

Problem: MM seemingly problematic for structuralism because

- Its objects are axiom systems, sentences, formulae, symbols,...
- These objects seem to be linguistic, hence *quasi-concrete* (Parsons).

Solution:

- Carefully distinguish between formal symbols (abstract, mathematical, criteria of identity relational) and linguistic objects representing them.
- Standard application of MM is to systems of linguistic objects, which is why it *seemed* to be about them.