Motivating Wittgenstein’s Perspective on Mathematical Sentences as Norms

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Outline of the Presentation

Axiomatics à la Hilbert

Implicit definitions as norms

From the axioms to the theorems

Summary
To introduce a Wittgensteinian idea, let’s start not with Wittgenstein himself but with Hilbert (and his axiomatic method).

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**Hilbert**

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Axioms as implicit definitions

Groundbreaking novelty of Hilbert’s “Foundations of Geometry” (1899):
- Axioms as (implicit) definitions of the concepts they contain
- No (external) criterion of truth for the axioms
- “[T]he role of intuition and observation is explicitly limited to motivation and is [merely] heuristic.” (Shapiro 2001)

Sharply contested by Frege, but Hilbert successful from a historical perspective (even in philosophy of mathematics, see structuralism)

Tait (2005): “The only conception of mathematics itself that I believe to be viable.”
Questions on Hilbertian axiomatics

If this is a (or the) standard modern approach to mathematics, philosophers should ask:

- What does it mean to treat an axiom as an implicit definition?
- What is the status of theorems derived in Hilbert-style axioms systems?
- Are they descriptions of anything?
- If so: of what and in which sense?

I shall argue: Propositions in Hilbert-style axioms systems are conceptual norms functioning as standards of what counts as using the concepts involved

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Wittgenstein on mathematical sentences as norms

From “Remarks on the Foundations of Mathematics”:

▶ [I]n mathematics we are convinced of grammatical propositions, so the expression, the result, of our being convinced is that we accept a rule. (RFM III, §26)

▶ What I am saying comes to this; that mathematics is normative. (RFM VII, §61)

▶ Mathematics forms a network of norms. (RFM VII, §67)

In this picture: propositions as knots of the web, proof as links between them.

Is this plausible for the axioms as implicit definitions? Arguably, yes!
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Ordinary definitions

Consider, as an example, the definition of “prime number”:

“A prime number is a natural number which has only two natural divisors: itself and 1.”

This proposition (used as a definition)

- is not used to describe anything mathematical.
- is adequate to existing usage (“descriptive definition”), but does not describe this usage.
- functions as a norm (standard of correctness) of what counts as using “prime number”.

⇒ For ordinary definitions the idea that they function as norms is plausible.
The axioms as implicit definitions

Consider

“Each natural number has a unique successor.” (Peano)

Arguably, when used as an implicit definition, this sentence plays a normative role in that it

- licences certain conceptual connections and rules out others (“the two successors of \( n \)”).
- must be accepted in order for the concepts involved to be used.
- partly constitutes what is meant by “Peano arithmetic”.

\( \Rightarrow \) For the axioms, treated as implicit definitions, Wittgenstein’s perspective is plausible as well.
“Let” and “by definition”

Wittgenstein in his Lectures:

“In a most crude way ... the difference between an experiential proposition and a mathematical proposition ... [is that] we can always affix to the mathematical proposition a formula like ‘by definition’.”

An alternative way of making the same point:

“Let each number have a unique successor.”

This formulation captures nicely how the axioms are used when treated as implicit definitions.

The “let” underlines the normativity.
All mathematical sentences as norms?

Motivated so far: Axioms (as implicit definitions) function as conceptual norms.

According to Wittgenstein: All (accepted) mathematical sentences are norms, even the derived ones.

Plausibility check: Is Fermat’s Last Theorem \((\forall n, x, y, z \in \mathbb{N}, n > 2 \ x^n + y^n \neq z^n)\) nothing but a mere conceptual norm?

Apparently natural idea: The axioms define the concepts. The theorems use them to describe the mathematical facts.
Can Ought imply Is?

However:

- If what the axioms do is constraining the use of concepts, how could we derive from them any truth about any realm of objects whatsoever?
- In other words: How could fact-stating sentences follow from mere conceptual norms?
- Crudely: How could one derive an Is from an Ought?
True: Theorems are not *primitive* norms in the same way as the axioms.

*But this does not make them any more descriptive!*

Someone not accepting a proven theorem *plays a different game*, just as someone not accepting an axiom.
Theorems as non-descriptive

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But this does not make them any more descriptive!

Someone not accepting a proven theorem plays a different game, just as someone not accepting an axiom.
An example

Consider three propositions which are equivalent in ZF set theory:

- Axiom of choice
- Zorn’s lemma
- Zermelo’s well-ordering theorem

Which one to use as an axiom?

As soon as equivalence shown: mode of use exactly equivalent.
Conclusion of the motivating line of thought

As far as the descriptive/normative distinction is concerned:

Axioms and theorems have the same mode of use.
Summary and conclusion

Wittgenstein’s (exotic) idea that mathematical sentences are used normatively has been motivated from Hilbert’s (mainstream) account of the axioms as implicit definitions:

▶ Definitions function as conceptual norms.
▶ This holds also for the axioms as implicit definitions.
▶ It plausibly extends to deductive consequences of the axioms (the theorems) as well.

⇒ Motivation accomplished.