Motivating Wittgenstein's Perspective on Mathematical Sentences as Norms

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Outline of the Presentation

Axiomatics à la Hilbert

Implicit definitions as norms

From the axioms to the theorems

Summary

See S. Friederich, Motivating Wittgenstein's perspective on mathematical sentences as norms, *Philosophia Mathematica* (3) 19:1-17 (2011).

To introduce a Wittgensteinian idea, let's start not with Wittgenstein himself but with

Hilbert

(and his axiomatic method).

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Axioms as implicit definitions

Groundbreaking novelty of Hilbert's "Foundations of Geometry" (1899):

- Axioms as (implicit) definitions of the concepts they contain
- no (external) criterion of truth for the axioms
- "[T]he role of intuition and observation is explicitly limited to motivation and is [merely] heuristic." (Shapiro 2001)

Sharply contested by Frege, but Hilbert successful from a historical perspective (even in philosophy of mathematics, see structuralism)

Tait (2005): "The only conception of mathematics itself that I believe to be viable."

Questions on Hilbertian axiomatics

If this is a (or *the*) standard modern approach to mathematics, philosophers should ask:

- What does it mean to treat an axiom as an implicit definition?
- What is the status of theorems derived in Hilbert-style axioms systems?
- Are they descriptions of anything?
- If so: of what and in which sense?

I shall argue: Propositions in Hilbert-style axioms systems are **conceptual norms functioning as standards of what counts as using the concepts involved**

This is a Wittgensteinian idea...

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Wittgenstein on mathematical sentences as norms

From “Remarks on the Foundations of Mathematics”:

- [I]n mathematics we are convinced of grammatical propositions,so the expression, the result, of our being convinced is that we accept a rule. (RFM III, §26)
- What I am saying comes to this; that mathematics is normative. (RFM VII, §61)
- Mathematics forms a network of norms. (RFM VII, §67)

In this picture: propositions as knots of the web, proof as links between them.

Is this plausible for the axioms as implicit definitions? Arguably, yes!

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Ordinary definitions

Consider, as an example, the definition of “prime number”:

“A *prime number* is a natural number which has only two natural divisors: itself and 1.”

This proposition (used as a definition)

- is not used to describe anything mathematical.
- is adequate to existing usage (“descriptive definition”), but does not *describe* this usage.
- functions as a norm (standard of correctness) of what counts as using “prime number”.

⇒ For ordinary definitions the idea that they function as norms is plausible.

The axioms as implicit definitions

Consider

"Each natural number has a unique successor." (Peano)

Arguably, when used as an implicit definition, this sentence plays a normative role in that it

- licences certain conceptual connections and rules out others ("the two successors of n").
- must be accepted in order for the concepts involved to be used.
- partly constitutes what is meant by "Peano arithmetic".

$\Rightarrow$ For the axioms, treated as implicit definitions, Wittgenstein's perspective is plausible as well.

“Let” and “by definition”

Wittgenstein in his *Lectures*:

“In a most crude way ... the difference between an experiential proposition and a mathematical proposition ... [is that] we can always affix to the mathematical proposition a formula like ‘ by definition’ .”

An alternative way of making the same point:

“*Let* each number have a unique successor.”

This formulation captures nicely how the axioms are used when treated as implicit definitions.

The “let” underlines the normativity.

All mathematical sentences as norms?

Motivated so far: Axioms (as implicit definitions) function as conceptual norms.

According to Wittgenstein: *All* (accepted) mathematial sentences are norms, even the derived ones.

Plausibility check: Is Fermat's Last Theorem ($\forall n, x, y, z \in \mathbb{N}$, $n > 2$ $x^n + y^n \neq z^n$) nothing but a mere conceptual norm?

Apparently natural idea: The axioms define the concepts. The theorems use them to describe the mathematical facts.

Can Ought imply Is?

However:

- If what the axioms do is constraining the use of concepts, how could we derive from them any truth about any realm of objects whatsoever?
- In other words: How could fact-stating sentences follow from mere conceptual norms?
- Crudely: How could one derive an Is from an Ought?

Theorems as non-descriptive

True: Theorems are not **primitive** norms in the same way as the axioms.

But this does not make them any more descriptive!

Someone not accepting a proven theorem *plays a different game*, just as someone not accepting an axiom.

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An example

Consider three propositions which are equivalent in ZF set theory:

- Axiom of choice
- Zorn's lemma
- Zermelo's well-ordering theorem

Which one to use as an axiom?

As soon as equivalence shown: mode of use exactly equivalent.

Conclusion of the motivating line of thought

As far as the descriptive/normative distinction is concerned:

Axioms and theorems have the same mode of use.

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Summary and conclusion

Wittgenstein's (exotic) idea that mathematical sentences are used normatively has been motivated from Hilbert's (mainstream) account of the axioms as implicit definitions:

- Definitions function as conceptual norms.
- This holds also for the axioms as implicit definitions.
- It plausibly extends to deductive consequences of the axioms (the theorems) as well.

$\Rightarrow$ Motivation accomplished.